# Creating and Determining the Contextuality of Quantum Error-Correcting Codes

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## Plain English Summary

Quantum computers can solve some problems that are very difficult for ordinary computers, but quantum information is extremely fragile. Small interactions with the environment can introduce errors and destroy a calculation. Quantum error correction protects information by spreading it across several physical qubits.

This project studied quantum error-correcting codes and their connection to *quantum contextuality*. Contextuality is a feature of quantum mechanics which shows that measurement results cannot always be explained by assuming that every measurement has one fixed value before it is performed.

The main outcome of the project was an understanding of how contextuality can appear directly inside the measurement structure of quantum error-correcting codes. In particular, subsystem stabilizer codes with at least two gauge qubits have a strongly contextual partial closure [2]. This creates a direct link between quantum error correction, the foundations of quantum mechanics, and fault-tolerant quantum computation.

## 1 Introduction

Quantum computers use qubits instead of classical bits. This allows them to use quantum effects such as superposition and entanglement. These effects can give quantum computers an advantage for problems such as factoring, simulation, optimisation and search.

The difficulty is that qubits are very sensitive to their environment. Noise can change a quantum state and cause errors. Without protection, these errors can build up and prevent a quantum computation from giving the correct answer. Quantum error correction is therefore an important part of building useful quantum computers [1].

This project investigated quantum error correction from the viewpoint of quantum contextuality. The aim was to understand what contextuality means, how it can be described mathematically, and how it appears in stabilizer and subsystem quantum error-correcting codes.

## 2 Why Quantum Error Correction is Needed

In classical computing, error correction can be quite simple. For example, a bit can be repeated three times:

$$0 \longrightarrow 000, \qquad 1 \longrightarrow 111.$$

If one bit changes, a majority vote can recover the original value. For example,

$$111 \longrightarrow 101 \longrightarrow 1.$$

The same idea cannot simply be used for a quantum state. An unknown quantum state cannot be copied perfectly. Also, directly measuring a qubit can disturb the state that we are trying to protect.

Quantum error correction solves this problem in a different way. Instead of making copies of the quantum state, one logical qubit is encoded across several physical qubits. The information is stored in correlations between these qubits rather than in one individual qubit.

This means that errors on some physical qubits can be detected without directly measuring the logical quantum information.

## 3 Stabilizer Codes

A major class of quantum error-correcting codes is the *stabilizer codes*. These codes use special multi-qubit measurements called stabilizers. Examples include

$$Z_1 Z_2$$

and

$$X_1X_2X_3X_4.$$

For a valid encoded state, stabilizer measurements normally give the result $+1$. If an error changes the state, some stabilizer measurements can instead give $-1$. The collection of these results is called the *error syndrome*.

The important point is that the stabilizers give information about errors without directly revealing the encoded logical information. This makes stabilizer measurements useful for fault-tolerant quantum computing [1].

# 4 Subsystem Stabilizer Codes

The project also considered *subsystem stabilizer codes*. These are a generalisation of ordinary stabilizer codes.

In a subsystem code, some encoded degrees of freedom contain the information that we want to protect. These are the *logical qubits*. Other degrees of freedom do not contain the important logical information and can be ignored. These are called *gauge qubits*.

Gauge qubits may initially seem unnecessary, but they can make the measurements required for error correction simpler. The extra gauge freedom can allow shorter and easier-to-measure check operators.

An important example is the Bacon–Shor code. The role of gauge qubits became especially important later in the project because the number of gauge qubits is directly related to strong contextuality.

# 5 Quantum Contextuality

The second main topic of the project was *quantum contextuality*.

In a classical description, it is natural to imagine that every measurable property already has a fixed value and that measurement simply reveals that value. Quantum mechanics does not always allow this picture [4].

The result of a quantum measurement can depend on the other compatible measurements considered with it. A collection of compatible measurements is called a *measurement context*.

Contextuality asks whether the outcomes from all of these different contexts can be explained using one fixed assignment of values.

If such a global assignment exists, the model can be described as non-contextual. If no suitable global assignment exists, the model is contextual.

# 6 The Hardy Example

Hardy's model was studied as a simple example of how contextuality appears.

Consider the quantum state

$$|\psi_H\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle + |00\rangle\right).$$

Alice and Bob can measure their qubits using different measurement bases. In the presentation these measurements were written as

$$A \equiv Z, \qquad A' \equiv X,$$

for Alice, and

$$B \equiv Z, \qquad B' \equiv X,$$

for Bob.

The same quantum state gives different possible outcomes depending on which measurement basis is chosen. The measurement contexts are

$$AB, \qquad AB', \qquad A'B, \qquad A'B'.$$

The possibilistic Hardy model records whether an outcome is possible or impossible in each context. A local outcome may be perfectly possible in one context but fail to extend to a single assignment covering all four measurements.

This gives a useful example of the basic idea behind contextuality: local measurement results can make sense individually while still failing to fit into one global classical picture.

# 7 The Sheaf-Theoretic Picture

The sheaf-theoretic framework gives a mathematical way of describing this local-to-global problem [3].

Each measurement context gives a piece of local information. We then ask whether all these local pieces can be joined, or "glued", together into one global assignment.

The main idea can be summarised as

$$\text{local assignments} \quad \longrightarrow \quad \text{global assignment?}$$

If the local assignments cannot be combined consistently, contextuality is present.

This also gives a way to describe *strong contextuality.* Strong contextuality is a stronger condition where no possible outcome can be part of a consistent global assignment.

The sheaf framework was useful in this project because it provides a common mathematical language for describing measurement contexts, local outcomes, global assignments and contextuality.

## 8 Contextuality as a Computational Resource

Contextuality is not only important for understanding the foundations of quantum mechanics. It is also connected with the computational power of quantum computers.

Howard *et al.* showed that contextuality is necessary for magic-state distillation [5]. Magic-state distillation is an important method for obtaining the resources needed for universal fault-tolerant quantum computation.

This means that contextuality can be viewed as a useful quantum resource rather than only as an unusual feature of measurement.

This observation motivated the main question of the project: if contextuality is useful for quantum computation, could it already be present inside the quantum error-correcting codes used to protect the computation?

## 9 Connecting Contextuality and Quantum Error Correction

Traditionally, quantum error correction and contextuality can appear to be separate topics. Quantum codes protect information, while contextuality describes the unusual behaviour of quantum measurements.

The work studied in this project shows that these topics are directly connected. The measurement structure of a quantum error-correcting code can itself be contextual [2].

The checks of a code can be treated as measurements. Checks which are compatible with each other form measurement contexts. We can then ask whether outcomes can be assigned to all the checks in a way that is consistent across every context.

This turns a quantum error-correcting code into a contextuality problem.

## 10 Partial Closure

An important idea in making this connection is the *partial closure* of the measurement set.

If two compatible checks can be measured together, then their product can also give information about the system. The partial closure includes measurements that can be obtained in this way from compatible measurements.

Therefore, instead of studying only the original checks, the partial closure studies the larger measurement structure that can be generated from them.

The contextuality of this partial closure can then be tested by asking whether its possible measurement outcomes admit a consistent global assignment.

## 11 Main Outcome of the Project

The most important result studied in this project concerns subsystem stabilizer codes.

Let $\mathcal{S}$ be a subsystem stabilizer code and consider its partial closure. The result is:

$$\boxed{\begin{array}{c}\text{At least two gauge qubits}\\ \Downarrow \\ \text{Strongly contextual partial closure}\end{array}}$$

More precisely, if the subsystem stabilizer code has two or more gauge qubits, its partial closure is strongly contextual. If it has fewer than two gauge qubits, the partial closure is non-contextual [2].

This is a particularly clear result because it connects a property of the error-correcting code directly to a property of quantum contextuality.

The number of gauge qubits is therefore not only important for the error-correcting structure of a subsystem code. It also tells us something about the nonclassical structure of its measurements.

The Bacon–Shor code provides an important example. With enough gauge degrees of freedom, its partial closure is strongly contextual.

The result also applies to code-switching protocols. The work studied in this project shows that code-switching schemes capable of achieving universal transversal gate sets are necessarily strongly contextual [2].

## 12 What the Project Shows

The project brought together several ideas which at first appear separate.

First, quantum information is fragile, so quantum error correction is required to protect it. Because quantum states cannot simply be copied, information must instead be encoded across several physical qubits.

Second, stabilizer measurements make it possible to detect errors without directly measuring the protected logical information. Subsystem codes extend this idea by introducing gauge qubits.

Third, contextuality shows that quantum measurement outcomes cannot always be understood using one fixed classical assignment. Hardy's model gives a simple example of this failure of a global description.

The sheaf framework then provides the mathematical connection between local measurement contexts and global assignments.

Finally, this framework can be applied directly to quantum error-correcting codes. The main result shows that the gauge structure of subsystem codes can force strong contextuality.

Therefore, the main outcome of the project is the connection

$$\boxed{\text{Quantum Error Correction} \quad \longleftrightarrow \quad \text{Quantum Contextuality.}}$$

The contextual behaviour is not necessarily something added later when a code is used in a quantum algorithm. It can already be present in the measurement structure of the code itself.

## 13 Why This Matters

This result is interesting from both a foundational and a practical point of view.

From a foundational point of view, it shows that contextuality is built into some of the structures used to protect quantum information. The same type of nonclassical behaviour studied in the foundations of quantum mechanics therefore appears naturally in quantum computing.

From a practical point of view, contextuality may provide another way to study fault-tolerant quantum codes. Since contextuality is already connected with resources needed for universal quantum computation [5], understanding which codes are contextual may help us better understand why particular fault-tolerant schemes are useful.

The result for subsystem codes is especially useful because it gives a simple structural condition: having at least two gauge qubits leads to strong contextuality in the partial closure.

## 14 Conclusion

This project studied the connection between quantum error correction and quantum contextuality.

The project began with the problem of protecting fragile quantum information. Classical repetition cannot simply be copied into quantum computing, so quantum information is instead encoded across several entangled physical qubits. Stabilizer measurements can then detect errors without directly measuring the logical information.

Subsystem stabilizer codes introduce gauge qubits, which can make error-correction measurements simpler. These gauge qubits also play an important role in contextuality.

Contextuality was studied using measurement contexts, Hardy's model and the sheaf-theoretic local-to-global picture. These ideas were then applied to the checks of quantum error-correcting codes.

The main outcome is that the partial closure of a subsystem stabilizer code is strongly contextual when the code has at least two gauge qubits, while codes with fewer gauge qubits are non-contextual in this setting [2].

This provides a clear link between the structure of quantum error-correcting codes and a fundamental non-classical feature of quantum mechanics. It also suggests that contextuality is an important part of understanding fault-tolerant and universal quantum computation.

## References


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