Minimally Primitive Graphs with a Non-cut Arc

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August 10, 2021

Primitive Matrices

Definition

A non-negative square matrix A is primitive if there is some positive integer k for which A^k is positive. The least such k is called the exponent of A.

Definition

Let A be a non-negative matrix. The graph of A is the directed graph $\Gamma(A)$ on vertex set $\{v_1, ..., v_n\}$, in which $v_i \to v_j$ is an arc if and only if $a_{ij} > 0$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure: A non-negative square matrix

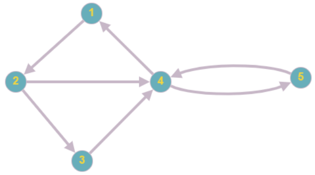


Figure: The graph of this matrix

Primitive Graphs

Lemma

Let A be a non-negative matrix with graph Γ. Let k be a positive integer. Then the (i,j)-entry of A^k is positive if and only if there is a walk of length k from v_i to v_j in Γ.

Corollary

A is primitive if and only if there exists a positive integer k with the property that whenever u and v are vertices of $\Gamma(A)$, there is a directed walk of a length k from u to v in $\Gamma(A)$. The least k for which this happens is the exponent of A.

Definition

A directed graph Γ is primitive if there is some positive integer k with the property that whenever u and v are vertices of Γ (not necessarily distinct) there is a walk of length k from u to v in Γ. The least k with this property is called the *exponent* of Γ.

Strongly Connected Graphs

Definition

A graph Γ is strongly connected if whenever u and v are vertices in Γ, there is a directed walk in Γ from u to v.

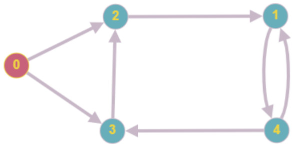


Figure: This graph is NOT strongly connected

Definition

A directed graph Γ is minimally strongly connected if Γ is strongly connected and $\Gamma \backslash e$ is not strongly connected for all $e \in E(\Gamma)$.

Minimally Primitive Graphs

Definition

A directed graph Γ is minimally primitive if Γ is primitive and $\Gamma \backslash e$ is not primitive for all $e \in E(\Gamma)$.

REMARK: A primitive graph that is minimally strongly connected, is also minimally primitive.

Definition

A non-cut arc is an arc in a strongly connected graph whose deletion leaves the graph strongly connected.

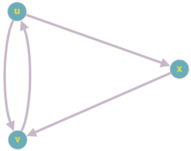


Figure: A minimally primitive graph with a non-cut arc

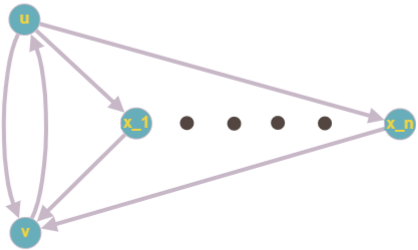
Minimally Primitive Graphs of exponent 5

Lemma

If Γ *is a minimally primitive directed graph then* $exp(\Gamma) > 4$.

Theorem

For $n \geq 3$ *let* G_n *denote the digraph on vertex set* $\{u, v, x_3, \ldots, x_n\}$ *with arcs* $u \to v$, $v \to u$, $u \to x_i$ *and* $x_i \to v$, *for* $3 \leq i \leq n$. *Then* G_n *is minimally primitive of exponent 5. Moreover if* $n \geq 3$ *and* G *is a directed graph of order* n *that is minimally primitive of exponent 5, then* $G \cong G_n$.



Bounds on the number of Arcs

Lemma

Let G be a minimally primitive graph with a non-cut arc and m(G) denote the number of arcs. Then :

$$n + 1 \leq m(G) \leq 2n - 2$$

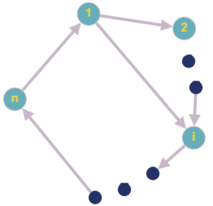


Figure: Exponent = $ab - 2a + b$

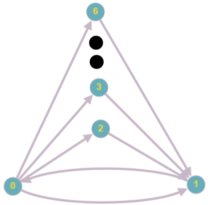


Figure: Exponent = 5

Minimally Primitive Graphs of Exponent 6 with non-cut arc

Lemma

Let G be a minimally primitive graph with non-cut arc e. If P is a nonempty set of primes then one of the following must hold

1. *e is in every circuit whose length is not divisible by any $p \in P$ OR*
2. *e is in every circuit whose length is not divisible by any $q \notin P$*

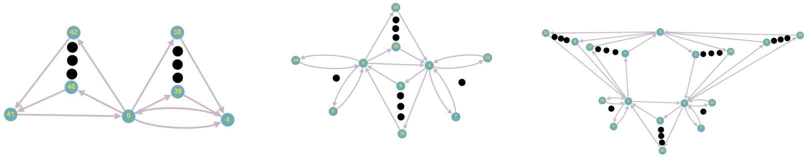
Lemma

Let G be a minimally primitive graph with exponent 6 and non-cut arc $u \to v$. Then $u \to v$ is in every circuit whose length is not divisible by 2 or 3.

Minimally Primitive Graphs of Exponent 6 with non-cut arc

Lemma

There exists a minimally primitive graphs of exponent 6 with a non-cut arc on $n \geq 5$ vertices.



Examples of exponent 6 minimally primitive graphs with a non-cut arc

Attempts to classify Exponent 6 graphs with a non-cut arc

Shortest Path